Homological stability of even and odd symplectic groups

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Homological stability

Group homology

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Fact: there is a *contractible* space EG on which G acts freely, and we denote $BG = EG/G$.

Note: $\pi_1(BG) = G$, universal cover of BG is contractible.

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Facts:

$H_*(G)$ gives algebraic information about G, for example $H_1(G)$ is the abelianization.

$H^*(G) =$ characteristic classes of principal G-bundles.

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Examples:

(i) $G_n = GL_n(F)$.
(ii) $G_n = S_n$, symmetric groups.
(iii) $G_n = \beta_n$, braid groups.
(iv) $G_g = Sp_{2g}(\mathbb{Z})$.
(v) $G_g = \Gamma_{g,1} = MCG(\Sigma_{g,1})$

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Questions:

1. Which families satisfy homological stability?
2. Can we compute the *stable homology*? i.e. compute $\operatorname{colim}_n H_d(G_n)$.

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Answer to question 1: Most families satisfy homological stability! All examples above do.

Thus, we can use homological stability to access $H_*(G)$ in some range!

New question: How good is homological stability? i.e find *the best* function f such that $H_d(G_n)$ stable for $d \leq f(n)$.

Results, version I

Motivation:

Theorem (Harer, Boldsen, Randal-Williams, Wahl, Galatius–Kupers–Randal-Williams, Harr–Vistrup–Wahl)
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Theorem (S.-Wahl)

The map

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$X \in C$ object.

$G_n = Aut(X^{\oplus n})$

$G_n \hookrightarrow G_{n+1}, f \mapsto f \oplus \mathrm{id}_X$.

Examples

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3. $\Gamma_{g,1}$: take $C =$ category with: objects $\Sigma_{g,1}$

   

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   $X = \Sigma_{1,1}$.

Proving homological stability

Key: under the assumptions of set up, one gets family of $(n-1)$-dimensional complexes $W_n(X)$. Informally, p-simplices are maps $X^{p+1} \to X^n$ in C.

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However, for $G_g = \Gamma_{g,1}$ optimal stability is $f(g) = (2g-2)/3$ and for $Sp_{2g}(\mathbb{Z})$ at least this good...

Question: how to prove it?

Half-speed stabilization

The work of Harr–Vistrup–Wahl

Idea: $X = \Sigma_{1,1}$ is not “one piece” but two!

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Then, get new family $G'_1 \hookrightarrow G'_2 \hookrightarrow G'_3 \hookrightarrow \dots$ with $G_g = G'_{2g+1}$.

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Informally: we add "half genus" surfaces to make family twice as big then get better stability.

If we add one handle to $\Sigma_{g,1}$ we get $\Sigma_{g,2}$.

Thus, $G'_{2g+2} = \Gamma_{g,2}$.

Have $\mathbf{\Gamma_{0,1}} \hookrightarrow \Gamma_{0,2} \hookrightarrow \mathbf{\Gamma_{1,1}} \hookrightarrow \Gamma_{1,2} \hookrightarrow \mathbf{\Gamma_{2,1}} \hookrightarrow \dots$

Geometry vs algebra

There is a canonical functor from category of surfaces and diffeomorphisms relative boundary to the category of skew-symmetric bilinear forms and isomorphisms:

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Problem: new algebraic family does NOT have stability! Also, stabilizing twice adds $(\mathbb{Z}^2, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix})$ not H.

Bi-marked surfaces

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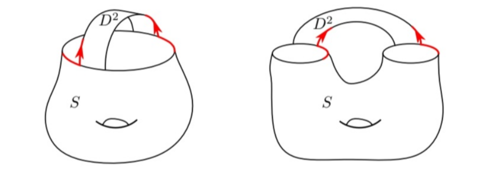
Answer (Harr-Vistrup-Wahl): Category M_2 with

Objects $(\Sigma, I_0 \sqcup I_1)$



Morphisms: diffeomorphisms fixing marking up to isotopy.

Monoidal structure $\#$: glue along marked intervals.

This solves problem of attaching handles!

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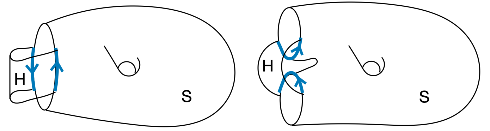
λ: skew symmetric form on M defined as follows:
$M = H_1(\Sigma, I_0 \sqcup I_1) \xleftarrow{\cong} H_1(\Sigma^+, D^1 \times I) \cong H_1(\Sigma^+)$, where
$\Sigma^+ = \Sigma \cup$ handle. Then, use intersection pairing on Σ^+.



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Answer: $\#$ induced from bimarked surfaces...

$(M_1, \lambda_1, \partial_1) \# (M_2, \lambda_2, \partial_2) = (M_1 \oplus M_2, ?, \partial_1 \oplus \partial_2)$

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Key: Cross-term has geometric meaning of classes crossing on extra handle we added to define intersection.

Fact: $\#$ has braiding in nice cases (but not symmetric)... braiding related to Bureau representations...

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We define the "odd" symplectic groups to be $Sp_{2g+1}(\mathbb{Z}) = Aut(X^{\#2g+2}) \cong Stab_{Sp_{2g}(\mathbb{Z})}(v)$.

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The family $G_n = Aut(X^{\#n})$ satisfies that $H_d(G_n) \to H_d(G_{n+1})$ is an iso if $d \leq \frac{n-2}{3}$.

The proof

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Idea: we can make sense of geometric objects algebraically!!

Example: an *arc* from b_0 to b_1= class $m \in M$ with $\partial(m) = 1$.

Non-separating arc= arc such that $\{m \bullet -, \partial\}$ unimodular.