Splitting Complexes

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Homological Stability

General Set-up

$X_0 \xrightarrow{s} X_1 \xrightarrow{s} X_2 \xrightarrow{s} \cdots$: sequence of spaces with maps between them.
Maps s are called "stabilization maps"

Question: Can we find a divergent function $f : \mathbb{N} \to \mathbb{N}$ such that $s_* : H_d(X_{n-1}) \to H_d(X_n)$ is an isomorphism for $d < f(n)$?

If so we say the family has *homological stability*

In practice: $f(n) = \lambda n + c$, λ= *slope* of the stability.

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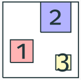
Classical approach

Studies case $X_n = BG_n$, G_n groups such that $\bigsqcup_n G_n$ is braided monoidal.

Based on studying connectivity of "destabilization complexes".

Homological Stability via E_k-algebras

Recall: E_k= little k-discs operad.



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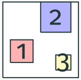

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New set-up

Assume $R := \bigsqcup_n X_n \in \mathrm{Top}^{\mathbb{N}}$ is a (graded) E_k-algebra.

Stabilization maps s induced by the E_k-product.

Example: In the classical set-up $R = \bigsqcup_n BG_n$ is a graded E_2-algebra.

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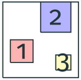 $\in E_2(3)$

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Cellular E_k-algebras approach (Galatius–Kupers–Randal-Williams)

Idea: Use the full E_k-structure to prove (better) homological stability results.

The cellular E_k-algebras machine

Notion of "cell attachment" in the category of graded E_k-algebras.

Thus get notion of cellular E_k-algebras.

There is a cellular approximation theorem.

Cells are bigraded: $D^{n,d}$= d-cell in grading n. The slope of a cell is d/n.

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Key:

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Main input: a lower bound for the slope of all cells.

Splitting complexes and a priory bounds on cells

Goal: To show that no cells of small slope are needed to build R cellularly.

Comes down to the high-connectivity of certain "*splitting complexes*".

Usually the technically most challenging part, but also very explicit!

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Given "object" x, its *splitting complex*, $S_\bullet(x)$ is a semisimplicial space with

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Goal: Estimate the connectivity of $S_\bullet(x)$ in terms of the grading of x.

Usually: If x is in grading x then $S_\bullet(x)$ is $(n-3)$-connected.

Example I: symmetric groups

Consider category of finite sets and bijections. Give it a symmetric monoidal structure by disjoint union. Its classifying space

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Alternative viewpoint:

A partition $\{1, \cdots, n\} = I_0 \sqcup \cdots \sqcup I_{p+1}$ is the same data as the *flag* $I_0 < I_0 \sqcup I_1 < \cdots I_0 \sqcup \cdots \sqcup I_p (< \{1, \cdots, n\})$.

From the flags viewpoint, face maps forget an element in the flag.

Thus, $S_\bullet(n)$= simplicial complex with

Vertices= non empty proper subsets of $\{1, \cdots, n\}$.

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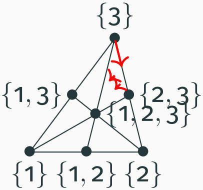

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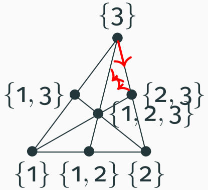

This is the barycentric subdivision of $\partial\Delta^{n-1}$.

Hence $S_\bullet(n) \cong S^{n-2}$ is $(n-3)$-connected.

Example II: configurations of points in the plane

Take X_n= unordered configurations of n points in the interior of I^2

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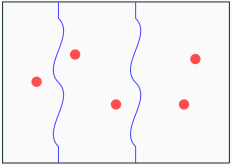
Take X_n= unordered configurations of n points in the interior of I^2

$R = \coprod_n X_n$ is an E_2-algebra.

Object=configuration of n points for a given natural n.

Splitting complex of a configuration x of n points has

$S_p(n)$= collection of partitions



Face maps: glue adjacent pieces in partition= forget walls

Fact: $S_\bullet(n)$ is $(n-3)$-connected.

Example III: symplectic groups

Consider category of skew-symmetric non-degenerate bilinear forms over a “nice” ring ($\mathbb{Z}$ or a field of characteristic $\neq 2$). Give it a symmetric monoidal structure by orthogonal direct sum. Its classifying space

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$S_p(n)$= ways of partitioning H^n into $p+2$ non-zero hyperbolics.

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Face maps: taking orthogonal direct sum of two adjacent pieces in partition.

Thus, $S_\bullet(n)$ is the nerve of the poset of hyperbolic subspaces of H^n, i.e. the “Tits complex”.

Fact (Looijenga–van der Kallen.): The Tits complex of H^n is $(n-3)$-connected.

Moduli spaces of manifolds

Fix a dimension N, usually $N \geq 3$.

Let $W_{g,1} := D^{2n} \# (S^n \times S^n)^{\# g}$.

Let X_g = collection of submanifolds of $I^{2N} \times \mathbb{R}^\infty$ which are diffeomorphic to $W_{g,1}$ and look standard near their boundary.

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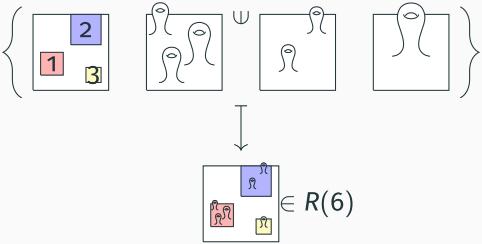
$R := \bigsqcup_g X_g$ is an E_{2N}-algebra:

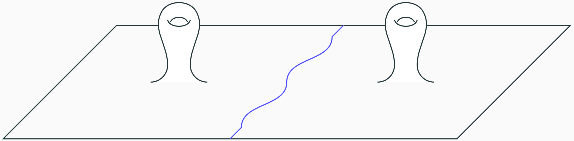
$$E_2(3) \times R(3) \times R(2) \times R(1)$$



The splitting complex of the moduli spaces of manifolds

Here object= submanifold $W \subset I^{2N} \times \mathbb{R}^\infty$ diffeomorphic to $W_{g,1}$ and standard near the boundary.

Thus, $S_p(g)$= ways of cutting a given $W \in X_g$ into $p+2$ objects of positive grading = space of $p+1$ "walls" in the manifold W.



$S_\bullet(g)$ is the nerve of the (topological) poset of walls.

Main theorem

Theorem (S. 2022)
The splitting complex $S_\bullet(g)$ *is* $(g-3)$*-connected for* $N \geq 3$ *odd.*

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The splitting complex $S_\bullet(g)$ is $(g-3)$-connected for $N \geq 3$ odd.

Application:

Theorem (S.2022) Let $N \geq 3$ be odd, then the stabilization maps

$$H_d(B\mathrm{Diff}_\partial(W_{g-1,1}); \mathbb{k}) \longrightarrow H_d(B\mathrm{Diff}_\partial(W_{g,1}); \mathbb{k})$$

are isomorphisms for

(i) $\mathbb{k} = \mathbb{Z}$ and $d \leq \frac{2g-4}{3}$ if $N = 3, 7$.
(ii) $\mathbb{k} = \mathbb{Z}[\frac{1}{2}]$ and $d \leq \frac{2g-7}{3}$ if $N \neq 3, 7$.
(iii) $\mathbb{k} = \mathbb{Q}$ and $d \leq \frac{2N-4}{2N-3}(g - 2 - \frac{2}{2N-4})$.

Proving the main theorem I

First Step: reduce to showing that the levelwise discretization $S_\bullet^\delta(g)$ has the same connectivity bound.

This step works quite generally, based on microfibrations.

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Second Step: covering the splitting complex and using a nerve theorem.

This step is a common trick employed in the literature when dealing with splitting complexes.

Key here: find an appropriate cover indexed by an easier poset.

For us the poset indexing cover is a generalization of the complex of non-separating arcs on a surface.

Proving the main theorem II

Third step: need to show high connectivity of the "arc complex".

Key idea: use homology to find an "algebraic model" of the complex.

By transversality and Whitney trick (hence $N \geq 3$) the high connectivity of the algebraic complex implies the one of the arc complex.

Proving the main theorem II

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Key idea: use homology to find an “algebraic model” of the complex.

By transversality and Whitney trick (hence $N \geq 3$) the high connectivity of the algebraic complex implies the one of the arc complex.

Final step: understanding the algebraic complex.

It is related to the complex of unimodular sequences, but impose an extra condition on their elements.