E_k-algebras and diffeomorphism groups

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Homology of diffeomorphism groups

$W =$ manifold with boundary

$\mathrm{Diff}_\partial(W) =$ diffeomorphisms of W fixing (pointwise) its boundary

$B\mathrm{Diff}_\partial(W) =$ classifying space of $\mathrm{Diff}_\partial(W)$

This classifies smooth fibre bundles with fibre W which are trivialized over ∂W

$H^*(B\mathrm{Diff}_\partial(W)) =$ characteristic classes of such bundles

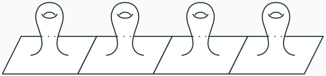
Convenient to discuss $H_*(B\mathrm{Diff}_\partial(W))$ instead.

Generalized surfaces

Focus our attention on a particular class of manifolds

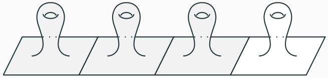
$$W_{g,1}^{2n} := D^{2n} \# (S^n \times S^n)^{\# g}$$

$W_{g,1}$ = boundary connected sum of g copies of $W_{1,1}$



Homological stability of diffeomorphism groups

Have inclusions $W_{g-1,1} \subset W_{g,1}$



which give stabilization maps

$$\mathrm{Diff}_\partial(W_{g-1,1}) \longrightarrow \mathrm{Diff}_\partial(W_{g,1})$$

Questions:

- Are these homology isomorphisms in a range of degrees?
- What is the stable homology? i.e. what is $\mathrm{colim}_g\, H_d(B\mathrm{Diff}_\partial(W_{g,1}))$?

What was known I: High dimensions

Let $n \geq 3$

• Homological stability for the family $\{\mathrm{Diff}_\partial(W_{g,1})\}_{g \geq 1}$: The stabilization maps

$$H_d(B\mathrm{Diff}_\partial(W_{g-1,1})) \longrightarrow H_d(B\mathrm{Diff}_\partial(W_{g,1}))$$

are isomorphisms for $d \leq \frac{g-4}{2}$ (Galatius–Randal-Williams 2012)

• The value of

$$\operatorname*{colim}_g H_d(B\mathrm{Diff}_\partial(W_{g,1}); \mathbb{Q})$$

is known (Galatius–Randal-Williams 2014).

What was known II: Surfaces

For $n = 1$ (surfaces)

$$\mathrm{Diff}_\partial(W_{g,1}) \xrightarrow{\simeq} \pi_0(\mathrm{Diff}_\partial(W_{g,1})) =: \Gamma_{g,1}$$

so one can study mapping class groups instead.

- Homological stability (of slope $1/3$) for $\{\Gamma_{g,1}\}_{g\geq 1}$ (Harer 1985).
- The stable (co)homology is known:

$$\lim_g H^*(B\Gamma_{g,1}; \mathbb{Q}) = \mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \cdots]$$

(Madsen–Weiss 2007)

What was known III: Further results on surfaces

- Improvements on the stability ranges for the $n = 1$ case (Ivanov, Boldsen, Randal-Williams): Best slope found was $2/3$.
- Using E_k-algebras the following results were shown:

(i) The stabilization maps

$$H_d(B\Gamma_{g-1,1}) \longrightarrow H_d(B\Gamma_{g,1})$$

are surjective for $d \leq \frac{2g-2}{3}$ and isomorphisms for $d \leq \frac{2g-4}{3}$

(ii) Some additional information: "Secondary stability" (which implies that $2/3$ slope is optimal).

(Galatius–Kupers–Randal Williams 2019)

What was known IV: Different approach to high dimensions

Theorem (Krannich 2019): Let $n \geq 3$, then the stabilization maps

$$H_d(B\mathrm{Diff}_\partial(W_{g-1,1});\mathbb{Q}) \longrightarrow H_d(B\mathrm{Diff}_\partial(W_{g,1});\mathbb{Q})$$

are isomorphisms for $d \leq \min\{g-3+c, 2n-5\}$, where $c = 0$ for n even and $c = 1$ for n odd.

Deduced from work of Berglund-Madsen (2012) on homological stability of "variations" $\{B\widetilde{\mathrm{Diff}}_\partial(W_{g,1})\}_g$ and $\{B\mathrm{Aut}_\partial(W_{g,1})\}_g$ of $\{B\mathrm{Diff}_\partial(W_{g,1})\}_g$.

Goal

Use E_k-algebras techniques to improve the ranges for $n \geq 3$.

• By Krannich expect a range of the form $d \leq g - 3 + c$ (with $\mathbb{Q}$ coefficients).

• Improvements on the stability range lead to the computation of new homology groups (and hence of characteristic classes).

• The improvements one can get from E_k-algebras techniques are of exploratory nature (better slopes, secondary phenomena etc).

The results

Statements (work in progress)

Theorem (in progress): Let $n \geq 3$, then the stabilization maps

$$H_d(BDiff_\partial(W_{g-1,1}); \mathbb{k}) \longrightarrow H_d(BDiff_\partial(W_{g,1}); \mathbb{k})$$

are isomorphisms for

(i) $\mathbb{k} = \mathbb{Z}$ and $d \leq \frac{2g-6}{3}$ if n is even.

(ii) $\mathbb{k} = \mathbb{Z}$ and $d \leq \frac{2g-4}{3}$ if $n = 3, 7$.

(iii) $\mathbb{k} = \mathbb{Z}[\frac{1}{2}]$ and $d \leq \frac{2g-7}{3}$ if n is odd, $n \neq 3, 7$.

(iv) $\mathbb{k} = \mathbb{Q}$ and $d \leq \frac{2n-4}{2n-3}(g - 2 - \frac{2}{2n-4})$ if n is odd. (This approaches Krannich's line when $n \longrightarrow \infty$)

The proof

Getting an E_{2n}-algebra

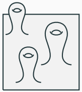
First step: view

$$R := \bigsqcup_{g \geq 1} B\mathrm{Diff}_\partial(W_{g,1})$$

as an ($\mathbb{N}$-graded) E_{2n}-algebra, so that stabilization maps arise from this E_{2n}-structure.

Point set model

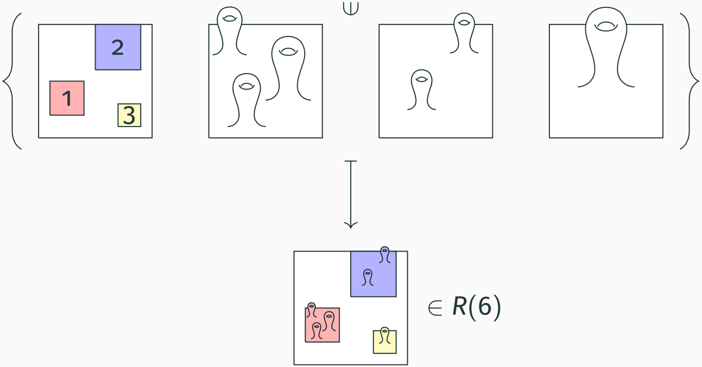
$R(g) := B\mathrm{Diff}_\partial(W_{g,1}) =$ space of submanifolds $W \subset I^{2n} \times \mathbb{R}^\infty$ diffeomorphic to $W_{g,1}$ and which look standard near the boundary $\partial W = \partial I^{2n} \times \{0\}$



$W \in R(3)$

The E_{2n}-algebra structure on R

$$\mathcal{C}_2(3) \times R(3) \times R(2) \times R(1)$$



$\in R(6)$

Technical remark: In the actual proof one considers a more convenient E_{2n-1}-algebra related to the above.

Notion of "cell attachment" in the category of E_k-algebras: given an E_k-algebra S (in $\mathbb{N}$-graded spaces) and a map $\varphi : \partial D^d \longrightarrow S(g)$ one can attach a bidegree (g, d) E_k-cell $E_k(D^{g,d})$ to get a new E_k-algebra $S \cup_{\varphi}^{E_k} E_k(D^{g,d})$.

Cellular E_k-algebras$=$ those constructed by a sequence of cell attachments starting from $\varnothing$.

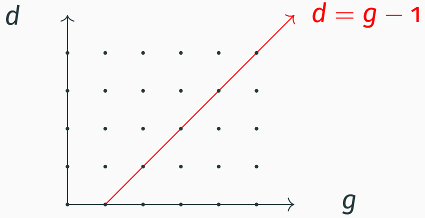
Cellular approximation theorem: If $R(0) = \varnothing$ then there is a cellular E_k-algebra A with a weak equivalence $A \xrightarrow{\simeq} R$.

Using the work of F.Cohen, the homology of cellular E_k-algebras is accessible.

Finding a (minimal) cellular approximation to R (I)

This is the second step in the proof and has two parts:

(i) Proving an a priori vanishing result: Show that there is a cellular approximation having no (g, d)-cells with $d < g - 1$.



(ii) Understanding a (minimal) cell structure for small values of (g, d).

"Adding cells of slope $d/g \geq \lambda$ does not destroy homological stability of slope $\leq \lambda$"

Finding a cellular approximation to R (II)

Vanishing result:

Comes down to the high-connectivity of certain ”splitting complexes”.

Usually the technically most challenging part.

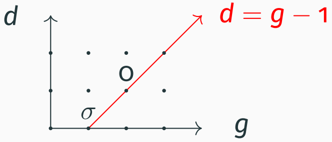
Cell structure for small degrees and genus:

Strategy is to access it by computing $H_d(B\mathrm{Diff}_\partial(W_{g,1}))$ for small (g, d).

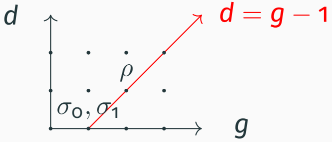
Cell computation integrally

Using the analysis of mapping class groups by Kreck (1979) we can access the cellular approximation up to degree 1.

n even:

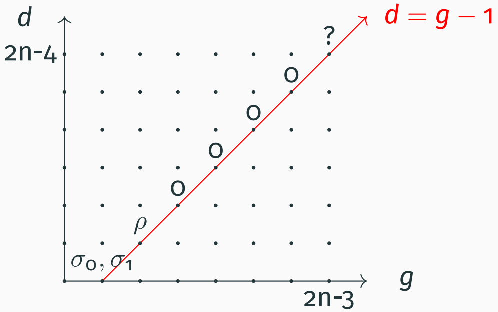


n odd:



Cell computation rationally for n odd

Using result by Krannich one shows that only 3 cells are needed on the critical line for $d \leq 2n-5$, so smallest slope of additional cells is $\geq \frac{2n-4}{2n-3}$.



Then one checks that the algebra consisting of just those 3 cells has rational stability of slope 1.