Mixed Tate motives over number fields and Goncharov's universality conjecture

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Joint work with Alexander Kupers and Daniil Rudenko

Introduction

Mixed Tate motives over number fields

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$\mathrm{MTM}_{\mathbb{Q}}(F) =$ category of (rational) mixed Tate motives over F.
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- This is a Tannakian tensor category, hence equivalent to

$$\mathrm{Comod}^{\mathrm{fd}}_{\mathscr{L}^{\mathrm{MTM}}(F)}(\mathrm{GrMod}_{\mathbb{Q}})$$

for $\mathscr{L}^{\mathrm{MTM}}(F)$ a graded Lie coalgebra, called *Lie coalgebra of (framed) mixed Tate motives*.

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Goal: To understand $\mathscr{L}^{\mathrm{MTM}}(F)$.

MTM and K-theory I

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Beilinson's formula (known for number fields):

$$K_{2n-i}^{(n)}(F)_{\mathbb{Q}} = \mathrm{Ext}^i_{\mathrm{MTM}_{\mathbb{Q}}(F)}(\mathbb{Q}(-n), \mathbb{Q}(0)).$$

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2. There is a cobracket $\delta^{\mathrm{MTM}} : \mathscr{L}^{\mathrm{MTM}}(F) \to \Lambda^2 \mathscr{L}^{\mathrm{MTM}}(F)$ preserving grading (no Koszul signs).
   Example:
   $\delta^{\mathrm{MTM}} : \mathscr{L}_4^{\mathrm{MTM}}(F) \to \mathscr{L}_1^{\mathrm{MTM}}(F) \otimes \mathscr{L}_3^{\mathrm{MTM}}(F) \oplus \Lambda^2 \mathscr{L}_2^{\mathrm{MTM}}(F)$.

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3. CE complex:

$$\mathscr{L}^{\mathrm{MTM}}(F) \to \Lambda^2\mathscr{L}^{\mathrm{MTM}}(F) \to \Lambda^3\mathscr{L}^{\mathrm{MTM}}(F) \to \cdots .$$

   E.g.
   $H^{\mathrm{CE}}_1(\mathscr{L}^{\mathrm{MTM}}(F))_n = \ker(\delta^{\mathrm{MTM}} : \mathscr{L}^{\mathrm{MTM}}_n(F) \to (\Lambda^2\mathscr{L}^{\mathrm{MTM}}(F))_n)$.

Algebraic K-theory of number fields

Clearly, $K_0(F)_{\mathbb{Q}} = \mathbb{Q}$, $K_1(F)_{\mathbb{Q}} = F^{\times}_{\mathbb{Q}} := F^{\times} \otimes \mathbb{Q}$.

Borel computed:

(i) $K_{2i}(F)_{\mathbb{Q}} = 0$ for $i \geq 1$.

(ii) $\dim_{\mathbb{Q}}(K_{2i-1}(F)_{\mathbb{Q}}) = \begin{cases} r_2 & \text{if } i \text{ even},\ i \geq 2 \\ r_1 + r_2 & \text{if } i \text{ odd},\ i \geq 3 \end{cases}.$

(iii) $K^{(i)}_{2i-1}(F)_{\mathbb{Q}} = K_{2i-1}(F)_{\mathbb{Q}}$ for $i \geq 1$.

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Consequence:

$$H_i^{\mathrm{CE}}(\mathscr{L}^{\mathrm{MTM}}(F))_n = \begin{cases} K_{2n-1}(F)_{\mathbb{Q}} & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

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Thus, $\mathscr{L}^{\mathrm{MTM}}(F) \cong \mathrm{cofree}_{\mathrm{coLie}}(\bigoplus_{n\geq 1} K_{2n-1}(F)_{\mathbb{Q}})$ (non canonically!)

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These are "motivic lifts" of *multiple polylogarithm* functions

$$\mathrm{Li}^{\mathrm{MTM}}_{n_1,\ldots,n_k}(a_1,\ldots,a_k) = \sum_{0<m_1<m_2<\cdots<m_k} \frac{a_1^{m_1}a_2^{m_2}\ldots a_k^{m_k}}{m_1^{n_1}m_2^{n_2}\ldots m_k^{n_k}}$$

for $|a_i| < 1$, $a_i \in \mathbb{C}$. Fact: They are periods of mixed Tate type.

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Universality conjecture (Goncharov): motivic multiple polylogs span $\mathscr{L}^{\mathrm{MTM}}(F)$ as a $\mathbb{Q}$-vector space. $\Rightarrow$ periods of mixed Tate type can be expressed in terms of multiple polylogs.

Results

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Theorem (Kupers–Rudenko–S.)

Let F be a number field. Then:

(i) $\mathscr{L}_n^{\mathrm{MTM}}(F)$ *is generated as a* $\mathbb{Q}$*-vector space by multiple polylogarithmic motives of weight* n*;*

(ii) *All relations between these follow from the ”decomposition relations”.*

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Note: δ^{MTM} explicitly described by Goncharov.

E.g. $\delta^{\mathrm{MTM}}(\mathrm{Li}_n^{\mathrm{MTM}}(a)) = \mathrm{Li}_{n-1}^{\mathrm{MTM}}(a) \otimes \mathrm{Li}_1^{\mathrm{MTM}}(1-a)$.

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Corollary

Goncharov's universality conjecture holds for number fields.

Motivic correlators

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They are "motivic lifts" of periods of mixed Tate type. E.g.

$$\mathrm{Cor}(0, \ldots, 0, a, 1) = -\int_{0 \le t_1 \le \ldots \le t_n \le 1} \frac{\gamma'(t_1)\, dt_1}{\gamma(t_1) - 1} \frac{\gamma'(t_2)\, dt_2}{\gamma(t_2)} \cdots \frac{\gamma'(t_n)\, dt_n}{\gamma(t_n)}$$

for γ a path from 0 to 1.

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Goncharov: formula for δ^{MTM} on motivic correlators:

$$\delta^{\mathrm{MTM}}(\mathrm{Cor}^{\mathrm{MTM}}(x_0, \ldots, x_n)) =$$

$$\sum_{j=0}^{n} \sum_{i=1}^{n-1} \mathrm{Cor}^{\mathrm{MTM}}(x_j, x_{j+1}, \ldots, x_{j+i}) \wedge \mathrm{Cor}^{\mathrm{MTM}}(x_j, x_{j+i+1}, \ldots, x_{j+n})$$

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$$\mathrm{Cor}^{\mathrm{MTM}}(x_0, \dots, x_n) - \mathrm{Cor}^{\mathrm{MTM}}(y_0, \dots, y_n) = \sum_{\iota = ((i_1, j_1), \dots, (i_n, j_n)) \in T(n)} \mathrm{sign}(\iota) \, \mathrm{Cor}^{\mathrm{MTM}} \left(0, \frac{x_{i_1} - x_{j_1}}{y_{i_1} - y_{j_1}}, \dots, \frac{x_{i_n} - x_{j_n}}{y_{i_n} - y_{j_n}} \right),$$

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where we omit terms with $y_{i_k} = y_{j_k}$ *for some* k*.* $T(n)$ *= certain collection of trees with vertices* $\{0, 1, \ldots, n\}$ *and ordering on set of edges.*

Results III

Set $T(n)$ determined inductively. E.g. $T(2)$ consists of tuples $(\{0,1\},\{0,2\}), (\{0,1\},\{1,2\})$, and $(\{0,2\},\{1,2\})$ with the signs

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For $n=2$, $\mathscr{L}_2^{\mathrm{MTM}}(F)\cong B_2(F)$ and decomposition relation $\iff$ 5-term relation.

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Similar results were know for $\mathbb{Z}$ by work of F. Brown.

The proof

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4. This algebra ”knows everything about K-theory”.

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Thus,

$$H^{E_\infty}_{n,d}(\mathbf{BGL}(F)_{\mathbb{Q}}) := H_d(\mathrm{indec}_{E_\infty}(\mathbf{BGL}(F)_{\mathbb{Q}})(n))$$

assemble into a shifted Lie coalgebra (cobracket has degree -1).

The Goncharov Lie coalgebra III

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Key insight: This object is "motivic" and it is also explicit

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Computation of cobracket is very technical. Based on finding a secondary cobracket in terms of the cobracket on St^∞.

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Presentation for $\mathscr{G}(F)$ + formula for $\delta \Rightarrow \exists$ *motivic realization* map $\mathscr{G}(F) \xrightarrow{r^{\mathrm{MTM}}} \mathscr{L}^{\mathrm{MTM}}(F)$ s.t. $\mathrm{Cor}^{\mathscr{G}}(x_0, \ldots, x_n) \mapsto \mathrm{Cor}^{\mathrm{MTM}}(x_0, \ldots, x_n)$.

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Recall: $\mathscr{L}^{\mathrm{MTM}}(F) \cong \mathrm{cofree}_{\mathrm{coLie}}(\bigoplus_{n\geq 1} K_{2n-1}(F)_{\mathbb{Q}})$ too! (non canonically).

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Result then follows since Borel regulator detects all K-theory of F.

Thank you for your attention

- A. Kupers, D. Rudenko, I. Sierra, *The Goncharov Lie coalgebra of a field*, arXiv:2606.23863, 2026.
- A. Kupers, D. Rudenko, I. Sierra, *Mixed Tate motives over number fields*, arXiv:2608.26392, 2026.